THEORY OF WAVES IN THE SHEAR FLOW OF A NONUNIFORM COMPRESSIBLE FLUID

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Early investigations of wave motion against a flow background are discussed in [1, 2]. An enormous number of papers have been devoted to the study of wave stability in nonuniform incompressible fluids. Here we cite only the review papers [1-5]. Recent analyses of wave phenomena have been particularly concerned with mode solutions i.e. discreet values of the frequencies, since this part of the spectrum is connected with hydrodynamical instability processes [6].

In [7] the effect of compressibility on flow stability was studied, and it was shown that in the linear approximation, compressibility leads to an increase in stability of flow. This discussion was developed further in [8], where it was shown that the stabilizing effect arises from the conversion of a definite amount of energy from the basic flow into work against the compressional forces.

In the present paper, wave motion in a nonuniform compressible fluid in the presence of shear flow is studied. We use the Hamiltonian approach [9-12] and develop further this formulation.

1. Basic Equations. Hamiltonian Formulation. We study the behavior of a nonuniform compressible fluid in the approximation of an isothermal atmosphere. This model is described by the following closed set of equations in the Euler representation: the Euler hydrodynamical equation

$$\dot{\mathbf{v}} + (\mathbf{v}\nabla)\mathbf{v} = -(1/\rho)\nabla p - \nabla\chi; \tag{1.1}$$

the equation of continuity

$$\dot{\rho} + \operatorname{div}\rho\mathbf{v} = 0; \tag{1.2}$$

the equation of conservation of entropy of the particles of the medium

$$\dot{\sigma} + (\mathbf{v}\nabla)\sigma = 0; \tag{1.3}$$

the relation for the internal energy, which plays the role of an equation of state

$$\mathscr{E} = \mathscr{E}(\rho, \sigma) = \mathscr{E}_0\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\exp\left[\frac{\gamma-1}{R}(\sigma-\sigma_0)\right]; \tag{1.4}$$

and the basic relation of thermodynamics

$$Td\sigma = d\mathscr{E} + pd(1/\rho). \tag{1.5}$$

In (1.1)-(1.5) the following notation is used: $\mathbf{v}$, hydrodynamical velocity; p, pressure; ρ, density; $\mathscr{E}$, σ, internal energy and entropy per unit mass; R, gas constant; γ, adiabatic exponent; and $\mathscr{E}_0$, ρ_0, σ_0, internal energy density, and entropy at z = 0.

Finally $\chi = gz$, where the z axis is taken along the vertical. In analogy with [9-11], the set of equations (1.1)-(1.4) can be represented in Hamiltonian form

$$\dot{\rho} = \delta E/\delta\varphi, \quad \dot{\varphi} = -\delta E/\delta\rho, \tag{1.6}$$

$$\dot{\sigma} = \delta E/\delta\lambda, \ \dot{\lambda} = -\delta E/\delta\sigma, \ \dot{\alpha} = \delta E/\delta\mu, \ \dot{\mu} = -\delta E/\delta\alpha,$$

where the energy of the medium has the form

$$E = \int d\mathbf{x}\cdot\rho\left\{\frac{\mathbf{v}^2}{2} + \mathscr{E} + \chi\right\} \quad (d\mathbf{x} = dxdydz),$$

and (ρ, φ), (σ, λ), (α, μ) are canonically conjugate pairs. Introduction of the Clebsch

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variables α, μ, which play the role of Lagrangian coordinates, is necessary for descriptions of vortex motion with complex topology, see [12-15], for example. Introductions of the additional pair of variables (α, μ) is necessary when it is not possible to describe the equilibrium state of the medium (i.e., a state where all physical quantities — velocity, entropy, density, etc. — are not functions of time) in terms of the variables (ρ, φ), (σ, λ). In our study it will be sufficient to consider only the pairs (ρ, φ) and (σ, λ); in this case the velocity can be represented in the form (see [11])

$$\mathbf{v} = \nabla\varphi - (\lambda/\rho)\nabla\sigma. \tag{1.7}$$

2. Shear Flow. We consider waves propagating against a steady-state dynamical background flow, denoted by subscript s. The velocity profile of the steady-state fluid flow is given in the form

$$\mathbf{v}_s = u(z)\mathbf{l},$$

where l is the unit vector along the x axis. A velocity profile of the above form is normally called shear flow. From the form of the external force field, we put $\rho_s = \rho_s(z)$, $\sigma_s = \sigma_s(z)$. Then the equations for $\dot{\varphi}$ and $\dot{\lambda}$ (see (1.6)) simplify considerably and can be reduced to the form

$$\dot{\varphi}_s + u^2/2 + gz + \gamma RT_s/(\gamma - 1) = 0, \tag{2.1}$$
$$\dot{\lambda}_s + u\partial\lambda_s/\partial x + \rho_s T_s = 0.$$

The solution to (2.1) is linear with respect to the time:

$$\varphi_s = ux - \left(\frac{u^2}{2} + \frac{\gamma RT_s}{\gamma-1} + gz\right)t,\ \lambda_s = -\rho_s T_s t + \rho_s\frac{\partial u}{\partial z}\left(\frac{\partial\sigma_s}{\partial z}\right)^{-1}(x - ut). \tag{2.2}$$

Substituting (2.2) into the expression for the velocity (1.7) and using the equation of state (1.4) we obtain the stratified density law

$$\rho_s(z) = \rho_0 \exp(-z/H)$$

and the entropy

$$\sigma_s(z) - \sigma_s(0) = Rz/H,$$

where $H = c_s^2/\gamma g$; and c_s is the adiabatic speed of sound. It is necessary to emphasize that φ_s and λ_s do not have a definite physical meaning, and are simply functions of the coordinates and the time. The first derivatives of these potentials do have physical significance; the derivatives $\partial\varphi/\partial x_i$ appear in the expression for the velocity (1.7) and $\partial\varphi/\partial t$ appears in the pressure.

The time dependence of the potentials φ_s and λ_s complicates the study of wave processes against the stationary state background. However, it can be shown (we omit the details) that in all orders of perturbation theory, one can if necessary eliminate the explicit time dependence of the velocity (and hence also that of the wave field Hamiltonian) using canonical transformations; thus nonlinear interactions can be considered to all orders. However, for our purposes, it is enough to eliminate the time dependence of the quadratic part of the Hamiltonian, which completely describes linear waves. To do this, a canonical transformation of the following form is carried out

$$\varphi = \varphi_1 + ux - \left(\frac{u^2}{2} + \frac{\gamma RT_s}{\gamma-1} + gz\right)t - \frac{R}{H}T_s\sigma_1 t + \frac{R}{H}\frac{\partial u}{\partial z}\left(\frac{\partial\sigma_s}{\partial z}\right)^{-1}(x - ut)\ \sigma_1, \tag{2.3}$$
$$\lambda = \frac{H}{R}\lambda_1 - \rho_1 T_s t + \rho_1\frac{\partial u}{\partial z}\left(\frac{\partial\sigma_s}{\partial z}\right)^{-1}(x - ut),\ \sigma = \frac{R}{H}\sigma_1 + \sigma_s,\ \rho = \rho_1,$$

where (ρ_1, φ_1) and (σ_1, λ_1) are the new canonical variables, which as before form canonical pairs. The proof that transformation (2.3) is canonical can be demonstrated in the usual way (see for example [16]), in our case we have

$$\int d\mathbf{x}\,\{\varphi_1\delta\rho_1 - \varphi\delta\rho + \lambda_1\delta\sigma_1 - \lambda\delta\sigma\} + DF_1 = (\mathscr{H}_1 - E)\,dt$$

where DF_1 is the total differential form of the generating functional F_1, and $\mathscr{H}_1$ is the new Hamiltonian. The generating functional is then calculated to be

$$F_1 = -t\int d\mathbf{x}\cdot\rho_1\left\{\frac{\mathbf{u}^2}{2} + \frac{\gamma RT_s}{\gamma-1} + dz + \frac{R}{H}T_s\sigma_1 + \frac{R}{H}\frac{\partial u}{\partial z}\left(\frac{\partial\sigma_s}{\partial z}\right)^{-1}u\sigma_1\right. + \int d\mathbf{x}\cdot\rho_1\left\{ux + \sigma_1 x\frac{\partial u}{\partial z}\right\},$$

and the Hamiltonian and velocity are given by

$$\mathscr{H}_1 = \int d\mathbf{x}\cdot\rho_1\left\{\frac{\mathbf{v}^2}{2} - \frac{\mathbf{u}^2}{2} + \mathscr{E} - \frac{\gamma R T_s}{\gamma-1} - \frac{R}{H}T_s\sigma_1 - \frac{R}{H}\frac{\partial u}{\partial z}\left(\frac{\partial\sigma_s}{\partial z}\right)^{-1}u\sigma_1\right\} \tag{2.4}$$

$$\mathbf{v} = u\mathbf{l} + \nabla\varphi_1 - \frac{\lambda_1}{\rho_1}\nabla\sigma_1 - \frac{\lambda_1}{\rho_1}\mathbf{n} + \sigma_1\nabla\left\{(x-ut)\frac{\partial u}{\partial z}\right\}$$

where n is the unit vector along the z axis. If we now write out Hamilton's equations in the new variables, calculate $\mathbf{v}$ and substitute this expression for $\mathbf{v}$ into the Euler equation (1.1), we reach an identity. This shows not only that the Hamiltonian structure of the equations is preserved (which is obvious; see [11]) but also that a steady-state dynamical regime (where the medium can be in a stationary state) can be realized in the medium. Earlier we formulated only the a priori assertion that such a state was available to the system (this is not always true in general), and a steady-state dynamical regime was only guaranteed when the consistency condition was satisfied.

3. Eigenvalue Problem. We write the Hamiltonian (2.4) in a more convenient form by performing the canonical transformation

$$\varphi = \varphi_1\rho_s^{1/2},\ \lambda = H\lambda_1\rho_s^{-1/2},\ \sigma = \frac{1}{H}\sigma_1\rho_s^{1/2},\ \rho = (\rho_1-\rho_s)\rho_s^{-1/2},\ \mathscr{H} = \mathscr{H}_1,$$

where ρ, φ, σ, λ are the new variables. We now expand the Hamiltonian in a power series in these variables; this yields the quadratic part of the Hamiltonian governing the wave-field which does not depend explicitly on time:

$$\mathscr{H}_{(2)} = \frac{1}{2}\int d\mathbf{x}\left\{\left[\left(\nabla + \frac{\mathbf{n}}{2H}\right)\varphi - \frac{\lambda}{H}\mathbf{n}\right]^2 - 2u\lambda\frac{\partial\sigma}{\partial x} + 2u\rho\frac{\partial\varphi}{\partial x} + 2H\frac{\partial u}{\partial z}\sigma\frac{\partial\varphi}{\partial x} + c_s^2\left[\rho^2 + 2\frac{\gamma-1}{\gamma}\rho\sigma + \frac{\gamma-1}{\gamma^2}\sigma^2\right]\right\}. \tag{3.1}$$

We apply a Fourier transform with respect to the horizontal coordinates $\vec{r} = \{x, y\}$ on (3.1) using the notation

$$\psi = \frac{1}{2\pi}\int\psi(\mathbf{k}_\perp, z)\,e^{i\mathbf{kr}}d\mathbf{k}_\perp,$$

where $\psi = \begin{Vmatrix}\varphi\\ \lambda\\ \rho\\ \sigma\end{Vmatrix}$; $\mathbf{k}_\perp = \{k\mathbf{l}, k\mathbf{m}\}$; m is the unit vector along the y axis.

With the help of the above notation, we write (3.1) in compact form (see also [17])

$$\mathscr{H}_{(2)} = \int dz d\mathbf{k}_\perp\psi^+(\mathbf{k}_\perp, z)\hat{\mathscr{H}}_0\psi(\mathbf{k}_\perp, z), \tag{3.2}$$

where

$$\hat{\mathscr{H}}_0 = \begin{Vmatrix} \mathbf{k}_\perp^2 + \frac{1}{4H^2} - \frac{\partial^2}{\partial z^2} - \frac{1}{H}\left(\frac{1}{2H} - \frac{\partial}{\partial z}\right) & -iu\mathbf{kl} & -H\frac{\partial u}{\partial z}i\mathbf{kl} \\ -\frac{1}{H}\left(\frac{1}{2H} + \frac{\partial}{\partial z}\right) & \frac{1}{H^2} & 0 & -iu\mathbf{kl} \\ iu\mathbf{kl} & 0 & c_s^2 & c_s^2\frac{\gamma-1}{\gamma} \\ H\frac{\partial u}{\partial z}i\mathbf{kl} & iu\mathbf{kl} & c_s^2\frac{\gamma-1}{\gamma} & c_s^2\frac{\gamma-1}{\gamma} \end{Vmatrix},$$

and the sign + denotes an Hermitian conjugate. In (3.2) we have used the relations

$$\int d\mathbf{x}(\nabla\varphi)^2 = -\int d\mathbf{x}(\varphi\Delta\varphi) + \int d\mathbf{x}\,\mathrm{div}(\varphi\nabla\varphi),$$

$$\int d\mathbf{x}\left(\lambda\frac{\partial\varphi}{\partial z}\right) = \int d\mathbf{x}\frac{\partial}{\partial z}(\varphi\lambda) - \int d\mathbf{x}\left(\varphi\frac{\partial\lambda}{\partial z}\right),$$

$$\mathbf{v}\to\mathbf{n}\ \text{for}\ z\to\pm\infty.$$

In matrix form, the equations of motion in the linear approximation take the form

$$\hat{A}\dot{\psi}(\mathbf{k}_\perp, z) = -i\frac{\delta\mathscr{H}_{(2)}}{\delta\psi^+(\mathbf{k}_\perp, z)}, \tag{3.3}$$

where

$$\hat{A} = \begin{Vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{Vmatrix}.$$

Taking the time Fourier transform of (3.3) we obtain the matrix equation

$$\begin{Vmatrix} \hat{h} & \hat{m} \\ \hat{n} & \hat{l} \end{Vmatrix} \cdot \begin{Vmatrix} \psi_1(\mathbf{k}_\perp, z) \\ \psi_2(\mathbf{k}_\perp, z) \end{Vmatrix} = \begin{Vmatrix} 0 & -i\Omega\hat{I} \\ i\Omega\hat{I} & 0 \end{Vmatrix} \cdot \begin{Vmatrix} \psi_1(\mathbf{k}_\perp, z) \\ \psi_2(\mathbf{k}_\perp, z) \end{Vmatrix}, \tag{3.4}$$

where

$$\hat{h} = \begin{Vmatrix} \mathbf{k}_\perp^2 + \frac{1}{4H^2} - \frac{\partial^2}{\partial z^2} & -\frac{1}{H}\left(\frac{1}{2H} - \frac{\partial}{\partial z}\right) \\ -\frac{1}{H}\left(\frac{1}{2H} + \frac{\partial}{\partial z}\right) & \frac{1}{H^2} \end{Vmatrix};$$

$$\hat{l} = c_s^2 \begin{Vmatrix} 1 & \frac{\gamma-1}{\gamma} \\ \frac{\gamma-1}{\gamma} & \frac{\gamma-1}{\gamma} \end{Vmatrix}; \quad \hat{n} = \hat{m}^+ = \begin{Vmatrix} 0 & 0 \\ H\frac{\partial u}{\partial z} i\mathbf{kl} & 0 \end{Vmatrix};$$

$$\hat{I} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}; \quad \Omega = \omega - u\mathbf{kl};$$

$$\psi_1(\mathbf{k}_\perp, z) = \begin{Vmatrix} \varphi(\mathbf{k}_\perp, z) \\ \lambda(\mathbf{k}_\perp, z) \end{Vmatrix}; \quad \psi_2(\mathbf{k}_\perp, z) = \begin{Vmatrix} \rho(\mathbf{k}_\perp, z) \\ \sigma(\mathbf{k}_\perp, z) \end{Vmatrix}.$$

Expanding out (3.4) we have the set of equations

$$\hat{h}\psi_1(\mathbf{k}_\perp, z) + \hat{m}\psi_2(\mathbf{k}_\perp, z) + i\Omega\hat{I}\psi_2(\mathbf{k}_\perp, z) = 0,$$
$$\hat{n}\psi_1(\mathbf{k}_\perp, z) + \hat{l}\psi_2(\mathbf{k}_\perp, z) + i\Omega\hat{I}\psi_1(\mathbf{k}_\perp, z) = 0,$$

which give a boundary-value problem for the potentials. For simplicity, we set up the problem for $\varphi(\mathbf{k}_\perp, z)$. After straightforward but tedious calculation, we find

$$\frac{\partial^2\varphi(\mathbf{k}_\perp, z)}{\partial z^2} + \frac{2}{\Omega}\left[\frac{\mathbf{kl}\frac{\partial u}{\partial z}N^2}{\Omega^2 - N^2}\right]\frac{\partial\varphi(\mathbf{k}_\perp, z)}{\partial z} + \frac{1}{\Omega^2 c_s^2}\left[\Omega^4 - c_s^2\left(\mathbf{k}_\perp^2 + \frac{1}{4H^2}\right)\Omega^2 - \right. \tag{3.5}$$

$$\left. - \frac{2c_s^2(\mathbf{kl})^2\left(\frac{\partial u}{\partial z}\right)^2\Omega^2}{\Omega^2 - N^2} - c_s^2(\mathbf{kl})\frac{\partial^2 u}{\partial z^2}\Omega + \frac{2c_s^2(\mathbf{kl})\frac{\partial u}{\partial z}\Gamma\Omega^3}{\Omega^2 - N^2} + N^2c_s^2\mathbf{k}_\perp^2\right]\varphi(\mathbf{k}_\perp, z) = 0.$$

Here $N^2 = -g\left(-\frac{1}{\rho_s}\frac{\partial\rho_s}{\partial z} + \frac{g}{c_s^2}\right) = \frac{g^2}{c_s^2}(\gamma - 1)$ is the square of the Vaisala frequency, and $\Gamma = \frac{1}{2\rho_s}\frac{\partial\rho_s}{\partial z} + \frac{g}{c_s^2}$ is the Eckart constant (see, for example, [18]).

The differential equation (3.5) is to be solved subject to the boundary conditions

$$\partial\varphi(\mathbf{k}_\perp, z)/\partial z \to 0, \ \varphi(\mathbf{k}_\perp, z) \to 0 \ (z \to \pm\infty),$$

specifying that wave motion is absent at infinity. The above differential equation has singularities at points z_i where the condition

$$\omega - u(z_1)\mathbf{kl} = 0, \ \omega - u(z_2)\mathbf{kl} + N = 0, \ \omega - u(z_3)\mathbf{kl} - N = 0$$

is satisfied. These singular points correspond to the so-called critical levels [18]. For a uniform medium we have $z_1 = z_2 = z_3$, i.e., the levels are degenerate. It should be pointed out that the treatment of wave processes in rotating reference frames also leads to two new critical levels [18]. Using the standard substitution

$$\varphi(\mathbf{k}_\perp, z) = \tilde{\varphi}(\mathbf{k}_\perp, z)\exp\left[-\int\frac{1}{\Omega}\frac{\mathbf{kl}\frac{\partial u}{\partial z}N^2}{\Omega^2 - N^2}dz\right]$$

we reduce (3.5) to the form

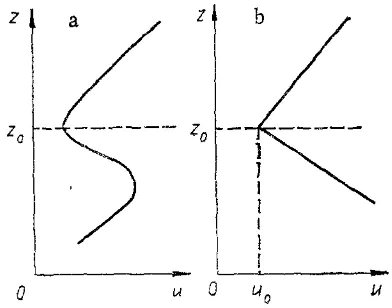

Fig. 1

$$\frac{\partial^2\tilde{\varphi}(\mathbf{k}_\perp, z)}{\partial z^2} + \frac{1}{\Omega^2 c_s^2}\left[\Omega^4 - c_s^2\left(\mathbf{k}_\perp^2 + \frac{1}{4H^2}\right)\Omega^2 - \frac{c_s^2(\mathbf{kl})^2\left(\frac{\partial u}{\partial z}\right)^2\Omega^2(2\Omega^2+N^2)}{(\Omega^2-N^2)^2} + \frac{2c_s^2\mathbf{kl}\frac{\partial u}{\partial z}\Gamma\Omega^3}{\Omega^2-N^2} + N^2c_s^2\mathbf{k}_\perp^2 - \frac{c_s^2\mathbf{kl}\frac{\partial^2 u}{\partial z^2}\Omega^3}{\Omega^2-N^2}\right]\tilde{\varphi}(\mathbf{k}_\perp, z) = 0. \tag{3.6}$$

The solution of the differential equation (3.6) is quite difficult for an arbitrary velocity profile. Therefore, we now limit our treatment to the case of a simple example, which apparently retains the basic features of more complicated situations. We will study only the case of discreet frequencies, since the mode solutions are the most interesting from the point of view of linear and nonlinear instabilities.

4. Specific Form of the Velocity Profile. One of the most typical forms of large-scale motion in atmospheres is shear flow. In Fig. 1a we show the typical dependence of the flow velocity on the height above the surface of the earth (the curve is taken from [18]). The shape of the curve about the extremum suggests an approximation of the form (Fig. 1b):

$$u(z) = u_0\left\{1 + (z - z_0)\left[\frac{1}{L_1}\theta(z - z_0) - \frac{1}{L_2}\theta(z_0 - z)\right]\right\},$$

where $\theta(t)$ is the Heaviside unit step function. The atmosphere is here considered to be unbounded with sufficiently large distance scales L_1 and L_2 (see below). The weak linear dependence of the velocity profile on the z coordinate suggests that in subsequent calculations we can ignore terms proportional to $\partial u/\partial z$ and $(\partial u/\partial z)^2$ if they do not contain delta-functions.

With these assumptions (3.6) simplifies considerably

$$\partial^2\tilde{\varphi}(\mathbf{k}_\perp, z)/\partial z^2 + 2[Q + \varkappa\delta(z)]\tilde{\varphi}(\mathbf{k}_\perp, z) = 0, \tag{4.1}$$

where

$$Q = \frac{1}{2\Omega^2 c_s^2}\left[\Omega^4 - c_s^2\left(\mathbf{k}_\perp^2 + \frac{1}{4H^2}\right)\Omega^2 + N^2c_s^2\mathbf{k}_\perp^2\right],$$

$$\varkappa = -\frac{\mathbf{kl}u_0}{2L}\frac{\Omega}{\Omega^2 - N^2}, \quad \frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}$$

and for simplicity we put $z_0 = 0$ and $u_0 > 0$. We also use relations of the form $\partial^2 u/\partial z^2 = u_0\delta(z)/L$ in the calculation. The conditions of applicability of (4.1) is the inequality $\varkappa L \ll 1$. Within our approximations, we can assume that $\Omega = \omega - \mathbf{ukl}$ is not a function of z. Then integrating (4.1) we obtain the jump condition

$$\left.\frac{\partial\tilde{\varphi}(\mathbf{k}_\perp, z)}{\partial z}\right|_{+0} - \left.\frac{\partial\tilde{\varphi}(\mathbf{k}_\perp, z)}{\partial z}\right|_{-0} = -2\varkappa\tilde{\varphi}(\mathbf{k}_\perp, 0).$$

Since we are studying the discreet frequency spectrum, we impose the condition $Q < 0$, $\varkappa > 0$. Under these assumptions, the solution to (4.1) can be represented in the form (see for example [19])

$$\tilde{\varphi}_j(\mathbf{k}_\perp, z) = B(\varkappa_j)\exp(-\varkappa_j|z|) \tag{4.2}$$

where the subscript j labels the wave mode and the coefficient $B(\varkappa_j)$ can be found from the normalization condition (see [17]). The solution corresponds to wave motion localized near the plane z = 0, with an amplitude that falls rapidly and exponentially with increasing distance from this plane. After substitution of the solution (4.2) into (4.1) and comparison of the coefficients, we obtain $Q_j = -\varkappa_j^2/2$ from which the dispersion equation is found in

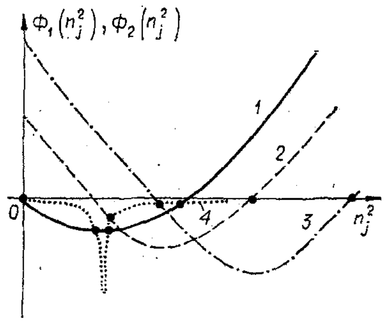

Fig. 2

explicit form

$$\frac{1}{\Omega_j^2 c_s^2}\left[\Omega_j^4 - c_s^2\left(\mathbf{k}_{\perp j}^2 + \frac{1}{4H^2}\right)\Omega_j^2 + N^2 c_s^2 \mathbf{k}_{\perp j}^2\right] = -\frac{(\mathbf{kl})_j^2 u_0^2 \Omega_j^2}{4L^2(\Omega_j^2 - N^2)^2}. \quad (4.3)$$

5. Analysis of the Dispersion Equation. The dispersion equation (4.3) is an eighth-order algebraic equation for the frequency ω (or Ω). In order to simplify the analysis of this equation, we introduce the dimensionless parameters

$$M_0 = u_0/c_s, \ \cos\alpha_j = (\mathbf{kl})_j/k_{\perp j}, \ n_j = H\Omega_j/c_s,$$
$$\mu = H/L, \ \nu_j = k_{\perp j}H.$$

In the new variables, (4.3) takes the simpler form

$$n_j^4 - \left(\nu_j^2 + \frac{1}{4}\right)n_j^2 + \frac{\gamma-1}{\gamma^2}\nu_j^2 = -\frac{1}{4}M_0^2\mu^2\nu_j^2\cos^2\alpha_j \frac{n_j^4}{\left[n_j^2 - \frac{\gamma-1}{\gamma^2}\right]^2}. \quad (5.1)$$

Below we will consider the constant $\varepsilon = \frac{1}{4}M_0^2\mu^2\nu_j^2\cos^2\alpha_j$ to be small. If we take $\mu \sim 1$, then the condition that ε be small becomes $\frac{1}{4}M_0^2\nu_j^2\cos^2\alpha_j \ll 1$. In subsonic flow when $M_0 \ll 1$, this does not impose a significant limitation on the wave number of propagating waves. We analyze the dispersion equation (5.1) graphically under the above assumptions. We plot graphs of the functions (see Fig. 2)

$$\Phi_1(n_j^2) = n_j^4 - \left(\nu_j^2 + \frac{1}{4}\right)n_j^2 + \frac{\gamma-1}{\gamma^2}\nu_j^2,$$
$$\Phi_2(n_j^2) = -\frac{1}{4}M_0^2\mu^2\nu_j^2\cos^2\alpha_j \frac{n_j^4}{\left[n_j^2 - \frac{\gamma-1}{\gamma^2}\right]^2}.$$

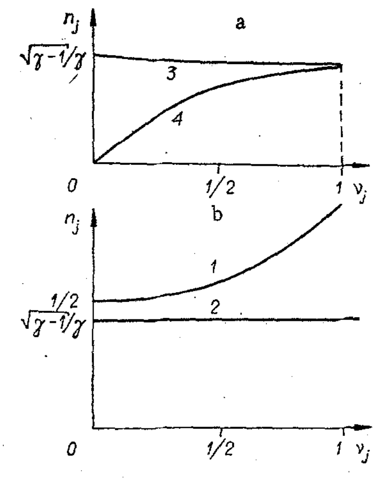

Fig. 3

Curves 1–3 show the behavior of $\Phi_1(n_j^2)$ for $\cos\alpha_j = 0.1$, $M_0 = 0.1$ and different fixed values of ν_j, where $\nu_{j1} < \nu_{j2} < \nu_{j3}$ (here the subscripts 1, 2, 3 label the different curves). Curve 4 shows the dependence of $\Phi_2(n_j^2)$. The number of points of intersection of the curves $\Phi_1(n_j^2)$ and $\Phi_2(n_j^2)$ for a fixed value of ν_j corresponds to the number of real roots of the dispersion equation (5.1). It follows from the graphical analysis that with increase of ν_j, the number of real roots decreases from four to two. The dispersion curve in a coordinate system coupled to the flow is shown in Fig. 3. Here Fig. 3a corresponds to waves propagating in the direction of the flow and Fig. 3b corresponds to propagation against the flow. From analysis of these dispersion diagrams, one can provisionally separate the waves into two types. The first type is an acoustic surface wave (curve 1); the second type are internal gravity waves propagating in the nonuniform flow (curves 2, 3, 4). The point where curves 3 and 4 join is $\nu_j = \nu^* = 1$ and is a bifurcation point. For $\nu_j > \nu^*$, a pair of complex conjugate values of n_j appear, which correspond to instability of waves of the second type. This instability is analogous to a centrifugal shear instability [7].

We now return to the relation

$$\varkappa_j = -\frac{(\mathbf{kl})\,u_0}{2L}\frac{\Omega_j}{\Omega_j^2 - N^2} > 0 \tag{5.2}$$

and apply it to wave motion for subsonic flow. Elementary analysis of (5.2) shows that curves 1 and 2 correspond to waves propagating opposite to the direction of flow, while 3 and 4 represent waves propagating along the flow. This refers to the propagation direction of the phase which, as is well known, does not in general coincide with the direction of the energy flux. We note that if waves of the types discussed above are excited in the system under consideration, their propagation will be waveguidelike with a selection effect. For example, the acoustic mode can propagate only in the direction opposite to the flow.

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FLOW PAST FORWARD-FACING SMALL STEP

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Uniform subsonic or supersonic laminar flow of a viscous fluid past a flat plate is considered. A small two-dimensional roughness element is present on the flat plate surface at a distance l from the leading edge. The solution to Navier–Stokes equations is developed for the case when the characteristic Reynolds number $Re_0 = \rho_0 u_0 l/\mu_0 = \varepsilon^{-2}$ tends to infinity (ρ_0, u_0, μ_0 are the density, velocity, and the coefficient of dynamic viscosity in the undisturbed free stream). In what follows, only nondimensional quantities will be used and for this purpose the reference quantities are: l for length, u_0 for velocity, ρ_0 for density, $\rho_0 u_0^2$ for pressure, u_0^2 for enthalpy, $\rho_0 u_0 l$ for the stream function, and μ_0 for the coefficient of dynamic viscosity. Systematic studies on the flow past small roughness over the surface of a body with characteristic transverse and longitudinal dimensions a and b ($\varepsilon^2 \ll a \lesssim \varepsilon$, $a \lesssim b \ll 1$) have been done in [1, 2], where, in particular, it has been shown that the flow near a roughness with $a \sim b \sim O(\varepsilon^{3/2})$ in the first approximation as $\varepsilon \to 0$, is described by Navier–Stokes equations for incompressible fluid, the velocity profiles and enthalpy in the external flow are sheared and the critical similarity parameter is the local Reynolds number $Re = \rho_w A a_1^2/\mu_w$ (the index w refers to the values at the flat plate surface in undisturbed boundary layer), A is the shear stress at the flat plate surface in undisturbed boundary layer, $a = \varepsilon^{3/2} a_1$, $a_1 \sim O(1)$. For Re, it is possible to obtain the following estimate [3]: $Re \sim Re_0^{1/2}(a/\varepsilon)^2$, from which it follows that as $a/\varepsilon \ll 1$ and $Re_0 \lesssim 10^6$ (i.e., for real and practically significant values of Re_0) the value of Re cannot exceed a few tens. Solutions to Navier–Stokes equations for the incompressible shear flow past a roughness on a body surface with $Re \lesssim 100$ are obtained in [4-6]. One of the distinctive features of these solutions is their existence at Re = 0 [5, 6], i.e., solutions of Navier–Stokes equations have been obtained for plane flows. Besides, even at Re = 0 separated zones have been observed in the flow field. The damping of disturbances far behind such roughness is also very typical and its study can be made with an analysis of the boundary layer equations along with the local condition for the interaction with the subsonic wall layer of the undisturbed boundary layer [7, 8].

It is useful to mention that the flow past roughness with $\varepsilon^{3/2} \ll a \sim b \ll \varepsilon$ in the first approximation as $\varepsilon \to 0$ is described by Euler equations for incompressible fluid with an external shear flow [1, 2].

Let there be a rectangular step on a flat plate with a characteristic height $a \sim O(\varepsilon^{3/2})$ and a characteristic length $\varepsilon^{3/2} \ll b \ll \varepsilon^{3/4}$. As shown in [1, 2], the flow past such a roughness is described by linearized incompressible boundary-layer equation with linearized local conditions for the interaction with the subsonic wall layer of the undisturbed boundary layer. It has been obtained in [8] that on the surface of such a step as one moves away from its face the disturbances in heat flux Δq and shear stress $\Delta\tau$ damp out at the following rate with respect to their values in the undisturbed boundary layer at the plate surface

$$\Delta q \sim \Delta\tau \sim x^{-1/3} \ (x \to \infty) \tag{2.1}$$

(i.e., damping of disturbances q and τ is very weak), and pressure disturbances $p < 0$ increase

$$|p| \sim x^{1/3} \ (x \to \infty). \tag{2.2}$$

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